

Tsemo, Aristide
 Department of Mathematics and Computer Sciences
 Ryerson University
 350, Victoria Street M1L 2V5, Toronto Canada
 tsemo58@yahoo.ca

Deformation of Homogeneous structures and Homotopy of symplectomorphisms groups.

Introduction.

Let (N, ω) a symplectic manifold, the group $Symp(N)$ of symplectomorphisms of (N, ω) acts transitively on it. Moreover, Banyaga has shown that $Symp(N)$ determines completely the symplectic structure of (N, ω) when N is compact. This motivates the study of the properties of $Symp(N)$ which must enable to understand the geometric properties of (N, ω) .

The beginning of the study of the homotopy properties of $Symp(N)$ has its origin in the theory of pseudo-holomorphic curves defined by Gromov, using this theory, Gromov has shown that the group of compactly supported symplectomorphisms in the interior of the symplectic 4-dimensional standard ball is contractible. Let $(S^2 \times S^2, \omega_0)$ be the product of two copies of the 2-dimensional sphere S^2 , endowed with the symplectic structure which is the product of the canonical volume form of S^2 . Gromov has also shown that $Symp(S^2 \times S^2)$ has the same homotopy type than $SO(3) \times SO(3)$.

The work of Gromov has been generalized by many authors, remark that if we endow $S^2 \times S^2$ with a symplectic form ω such that the volume of the fiber of the fibration $S^2 \times S^2 \rightarrow S^2$ is not the volume of its base, the homotopy of $Symp(S^2 \times S^2, \omega)$ can be different than the homotopy of a finite dimensional Lie group. This phenomenon has been firstly observed by Abreu, and completely studied by McDuff and Abreu. Their study use the theory of pseudo-holomorphic curves, and stratification of the space of pseudo-convex structures.

In his thesis Pinsonnault has studied the homotopy type of the one point blow-up N of $S^2 \times S^2$, if the volume of the fiber and base of the fibration $S^2 \times S^2$ are equal, he has shown that $Symp(N)$ has the homotopy type of a 2-dimensional torus. Joseph Coffey has used the decomposition of a symplectic 4-dimensional manifold to relate the homotopy type of the group of symplectomorphisms $Symp(N)$ of a 4-dimensional manifold to a configuration space, more precisely Paul Biran has shown that we can remove a simplex D on a 4-dimensional symplectic manifold in such a way that the resulting space is a disc bundle over a symplectic surface, the group $Symp(N)$ is then an extension of its subgroup which preserves D by the orbit of D by the group $Symp(N)$. This approach must be related to the work of Lalonde and Pinsonnault who

have studied the relations of the space of embedding of a symplectic ball with a given radius in a symplectic 4-dimensional manifold and $Symp(N)$.

As remarked Pinsonnault in his thesis, the calculation of the homotopy type of the group of symplectomorphisms of $S^2 \times S^2$ and its blow-up is possible because the topology of these manifolds is very simple, and there exists a good knowledge of the theory of pseudo-holomorphic curves of these manifolds. Another simple example of symplectic 4-dimensional manifold is the 4-dimensional torus N^4 , the study of $Symp(N^4)$ is hardly tractable using the theory of pseudo-holomorphic curves since $H^2(N^4, \mathbb{R})$ is a 6-dimensional real vector space.

The purpose of this paper is to describe the homotopy type of $Symp(N^4)$ using new ideas. The torus N^4 is the quotient of the affine space \mathbb{R}^4 by 4-translations which preserve the standard symplectic form. Let D' be the boundary of a fundamental domain of this action. It projects to N^4 to define a union D of 3-dimensional torus. Our approach may be related to the work of Joseph Coffey and Lalonde-Pinsonnault, we remark that the group $Symp(N^4)$ is the total space of a fibration whose base space is the orbit of D under $Symp(N^4)$ and fiber the group of symplectomorphisms which preserve D .

To determine the homotopy type of $Symp(N^4)$, one needs to determine the homotopy type of the space $N(D)$ of orbits of D under $Symp(N)$. This is performed using the theory of deformation of homogeneous structures as it is described by W.Goldman. We show that $N(D)$ is contractible.

Decomposition of symplectic manifolds and group of symplectomorphism.

The purpose of this paragraph is to define a notion of canonical decomposition of a compact symplectic 4-dimensional manifold, and to show that the connected component of the group of Hamiltonian diffeomorphisms which preserve such a configuration is contractible.

Definition

Let (N, ω) be a compact 4-dimensional symplectic manifold, a configuration D of (N, ω) is a finite union D of compact 3-dimensional submanifolds D_1, \dots, D_p which satisfy the following conditions:

The manifold $N - D$ is symplectomorphic to the 4-dimensional symplectic standard ball or to a polydisc endowed with the restriction of the standard symplectic form of \mathbb{R}^4 , D is connected.

The restriction of ω defines on D_i a contact structure such that the fibers of the characteristic foliation are circles, D_i is the total space of a bundle whose leaves are the characteristic leaves, and whose base space is an intersection $D_i \cap D_j$. The fibers of the characteristic foliation of D_i intersects at least two different cells.

The intersection $D_{ij} = D_i \cap D_j$ is either a symplectic 2-dimensional submanifold of (N, ω) , or a Lagrangian 2-dimensional submanifold stable by the characteristic foliations.

Suppose that D_{ij} is a symplectic submanifold and consider h_t a path of Hamiltonian diffeomorphisms of D_{ij} such that the restriction of h_t to $D_{i_1 i_2 i_3} = D_{i_1} \cap D_i \cap D_j$ is the identity if $D_{i_1 i_2 i_3}$ is a circle, then we can extend h_t to a path of Hamiltonian diffeomorphisms h'_t of N in such a way that it preserves the configuration, and its restriction to the cells D_l such that $D_l \cap D_{ij}$ is either a circle, or is empty is the identity. The restriction of h'_t to a symplectic surface $D_{i'j'}$ different of D_{ij} is the identity.

The configuration defines on N the structure of a CW-complex with one 4-dimensional cell.

An example of a configuration is the 4-dimensional torus N^4 . It is the product of 2-dimensional torus N_2 and N'_2 . N^4 is endowed with the symplectic form which is the product of the standard symplectic forms of N_1 and N_2 . Let l_1, l_2 , be the two curves of N_2 parallel in respect to the flat structure which generate $\pi_1(N_2)$, and l_3, l_4 , the curves of N'_2 which satisfy the same properties. The configuration is the union of $N_2 \times l_3, N_2 \times l_4, N'_2 \times l_1, N'_2 \times l_2$. Each function $H : N_2 \rightarrow \mathbb{R}$, which is constant on l_1 and l_2 can be extended to $N_2 \times N'_2$ by the projection $N_2 \times N'_2 \rightarrow N_2$, and it preserves the configuration.

We denote by $Symp(N, D)$ the group of symplectomorphisms of (N, ω) which preserve D , $Ham(N, D)$ its subgroup of Hamiltonian symplectomorphisms, $Symp_{Id}(N, D)$ is the subgroup of $Symp(N, D)$ whose restriction to D is the identity, and $Ham_{Id}(N, D)$ is the subgroup of Hamiltonian diffeomorphisms of $Symp_{Id}(N, D)$.

Proposition *The connected component $Ham_{Id}(N, D)_0$ of the group $Ham_{Id}(N, D)$ is contractible.*

A theorem of Weinstein asserts that there exists a neighborhood U_{Id} of the identity in $Symp(N)$ which is contractible. Without restricting the generality, we can suppose that the image of U_{Id} by the flux map is a contractible open subset. Thus we can suppose that $Ham(N) \cap U_{Id}$ is contractible, and the existence of a continuous map $Ham(N) \cap U_{Id}$ to the space of exact time dependent 1-forms, $h \rightarrow dH^h$, where $H^h : N \rightarrow \mathbb{R}$ is a time dependent differentiable function such that h is the value at 1 of the flow generated by dH .

Without restricting the generality, we can suppose that for every element $h \in Ham_{Id}(N, D)_0$, there exists a neighborhood V of D such that the restriction of h to V is the restriction of an element of U_{Id} , h' to V which coincide with h on an open neighborhood U which contains V , and $h(V) \subset U_1 \subset U$, where U_1 is open. The existence of h' can be shown using a cut function. We can also suppose that $N - V, N - U$ are polydiscs. Let f be a cut function such that the restriction of f to U_1 is 1 and the restriction of f to $N - U$ is zero, we define a map $\Phi_t, h \rightarrow \psi_t^{-1} \circ h$, where ψ_t is the flow generated by $fH^{h'}$, $\Phi_0(h) = h$, $\Phi_1(h)$ is an Hamiltonian map whose support is contained in the interior of $N - V$. Let $Ham_{Id}(N, V, U_1, U, D)$ be the subset of $Ham_{Id}(N, D)_0$ such that the restriction of h to V is the restriction of an element h' of U_{Id} , and $h(V) \subset U_1$, the restrictions of h and h' to U coincide. A result of Gromov says that the group of symplectomorphisms whose supports are in the interior

of $N - V$ is contractible. The map Φ_t implies that $Ham_{Id}(N, V, U_1, U, D)$ is contractible.

Let $(V_n)_{n \in \mathbb{N}}$, be a family of open subsets which contain D such that V_{n+1} is contained in V_n , $\cap_{n \in \mathbb{N}} V_n = D$. Consider the families of open subsets U_n and U_1^n , such that $V_n \subset U_1^n \subset U_n$, we suppose that $\cap_{n \in \mathbb{N}} U_n = \cap_{n \in \mathbb{N}} U_1^n = D$, $U_1^{n+1} \subset U_1^n$, $U_{n+1} \subset U_n$. Then $Ham_{Id}(N, D)_0$ is the limit of $Ham_{Id}(N, V_{n+1}, U_1^{n+1}, U_{n+1}, D)$ when the previous expression has a sense. We suppose that $N - V_n$ is a polydisc. Since V_{n+1} is contained in the interior of V_n , we deduce that $Ham_{Id}(N, V_n, U_1^n, U_n, D)$ is contained in the interior of $Ham_{Id}(N, V_{n+1}, U_1^{n+1}, U_{n+1}, D)$. This implies that $Ham_{Id}(N, D)_0$ is contractible.

Proposition. *The group $Ham(N, D)_0$ is contractible.*

Proof. Let L_D be the restriction to D of the subgroup $Ham'(N, D)$ of $Ham(N, D)_0$ whose restriction to $D_{i_1 i_2 i_3}$ is the identity if $D_{i_1 i_2 i_3}$ is a curve. If D_{ij} is a symplectic submanifold, we denote by L_{ij} the group of Hamiltonian diffeomorphisms of D_{ij} whose restriction to $D_{i_1 i_2 i_3}$ is the identity, if $D_{i_1 i_2 i_3}$ is a circle, we denote by L'_D the product of the groups L'_{ij} . The extension property verified by the configuration implies that the restriction map $L_D \rightarrow L'_D$ is a fibration. The fiber of this fibration is contractible, since the gauge group of the fibration induced by the characteristic foliation is contractible, since each fibers intersects two different cells, and a connected subgroup of diffeomorphisms of an interval is contractible. Since L'_D is contractible, we deduce that L_D is contractible.

Let L_D^1 be the restriction of $Ham(N, D)_0$ to D , and L_D^2 the restriction of $Ham(N, D)_0$ to the circles $D_{i_1 i_2 i_3}$, we have a fibration $L_D^1 \rightarrow L_D^2$ whose fiber is L_D , since L_D is contractible, and L_D^2 is contractible since a connected group of diffeomorphism of an interval is contractible, we deduce that L_D^1 is a contractible. The fiber of the fibration $Ham(N, D)_0 \rightarrow L_D^1$ is $Ham_{Id}(N, D)_0$ which is contractible. This implies that $Ham(N, D)_0$ is contractible.

Isotopy of configurations and homogeneous structures.

To determine the homotopy of $Ham(N)$, we have to determine the homotopy of the orbit of D under the action of $Ham(N)$. To solve this problem, we consider the situation when N is a locally homogeneous manifold.

Definition Let V be a differentiable manifold, and H a Lie subgroup acting on V , the action of H satisfies the unique extension property, if and only if two elements h_1, h_2 of H which coincide on an open subset of V , coincide on V .

Definition Let (V, H) be a n -dimensional manifold endowed with the action of a Lie group which satisfies the unique extension property. A manifold N is endowed with a (V, H) (homogeneous) structure, if and only if there exists a V -atlas $(U_i, \phi_i)_{i \in I}$ of N , such that $\phi_i : U_i \rightarrow V$, and $\phi_j \circ \phi_i^{-1}$ is the restriction of the action of an element h_{ij} of H on $\phi_i(U_i \cap U_j)$.

Let N be a manifold endowed with a (V, H) homogeneous structure, and \hat{N} be the universal cover N , \hat{N} inherits from N a structure (the pull-back)

of a (V, H) manifold defined by a local diffeomorphism $D_N : \hat{N} \rightarrow V$ called the developing map. This developing map gives rise to a representation $h_N : \pi_1(N) \rightarrow H$ defined by the following commutative diagram:

$$\begin{array}{ccc} \hat{N} & \xrightarrow{d} & \hat{N} \\ \downarrow D_N & & \downarrow D_N \\ V & \xrightarrow{h_N(d)} & V \end{array}$$

called the holonomy representation.

Examples of homogeneous manifolds are:

Affine manifolds, here V is the affine space \mathbb{R}^n , and H is $Aff(\mathbb{R}^n)$ the group of affine transformations.

Projective manifolds, here V is either $P^n(\mathbb{R})$, or $P^n(\mathbb{C})$, and H is the group of projective transformations.

Let N be a manifold endowed with a (V, H) structure, we denote by $D(N, V, H)$ the space of (H, N) structures defined on N . To each element $[h_N]$ of $D(N, V, H)$, we can associate the following flat bundle V_{h_N} : let (U_i, ϕ_i) be the atlas which defines the (V, H) -structure of N . The total space of V_{h_N} is the quotient of $\hat{N} \times V$ by the action of $\pi_1(N)$ defined on the first factor by the Deck transformations, and on the second factor by the holonomy h_N of the (V, H) structure. The following result has been shown in Goldman:

Theorem.

Let N be a compact manifold endowed with a (V, H) structure, then the map $D(N, V, H) \rightarrow Rep(\pi_1(N), H)$ which assigns to each (V, H) structure its holonomy representation is an open map, two nearby (V, H) structures with the same holonomy are isotopic.

Suppose now that the (V, H) -manifold N is endowed with a symplectic form ω , and there exists a configuration D of N such that $N - D$ is symplectomorphic to the standard ball, or the standard polydisc. The idea that will be used is to study the deformation space of D under the action of the group of symplectomorphisms, using the deformation theory of locally homogeneous structures developed in Goldman. Let ϕ be a symplectomorphism of (N, ω) the symplectomorphism ϕ acts on $D(N, V, H)$ as follows: let $(U_i, \phi_i)_{i \in I}$ be an atlas which defines the (V, H) structure, then $(\phi(U_i), \phi_{i'} \circ \phi)$ defines a (V, H) structure $\phi([h_N])$. (without restricting the generality, we suppose that there exists $i' \in I$ such that $\phi(U_i)$ is contained in the domain of a chart $U_{i'}$). The affine structure of h_N is defined by the projection on the trivialization $U_i \times V$, on the second factor. It can be defined also by the following section l_{h_N} of V_{h_N} : we define the restriction $l_{h_N}^i$ of l_{h_N} to U_i to be the map defined by $l_{h_N}^i(u) = (u, \phi_i(u))$. The affine structure of $\phi(h_N)$ is defined by a section of V_{h_N} transverse to the horizontal foliation after the identification between V_{h_N} and $V_{\phi(h_N)}$. Recall that the horizontal foliation of V_{h_N} is the foliation whose leaves are the projections of the submanifolds $\hat{N} \times \{u\}$ of $\hat{N} \times V$ by the covering map $\hat{N} \times V \rightarrow V_{h_N}$.

Suppose that (V, H) is $(\mathbb{R}^n, \text{Aff}(\mathbb{R}^n))$, in this situation, the bundle V_{h_N} is the affine vector bundle defined on N , it is associated to a principal bundle defined on N , which is the bundle of affine frames. Recall that an affine manifold is also defined by a connection whose curvature and torsion forms vanish identically, this connection induces on the bundle of affine frames a connection a la Erhesmann, the horizontal distribution which defines the Erhesmann connection induced the horizontal foliation of h_{h_N} .

The action of $\hat{\text{Symp}}(N)_0$ on V_{h_N} .

We are going to define an action of the connected component of the universal cover $\hat{\text{Symp}}(N)_0$ of the group of symplectomorphisms $\text{Symp}(N)$ of (N, ω) on V_{h_N} . We suppose that our (V, H) structure is complete, or more generally, we suppose that the developing map is a covering map, this is equivalent to saying that N is the quotient of an open subset U_N of V by a discrete subgroup of H .

We define $V_{h_N}^1$ to be the U_N -bundle defined on N , by making the quotient of $\hat{N} \times U_N$ by the action of $\pi_1(N)$ defined on the first factor by the deck transformations, and on the second factor by the action of the holonomy.

Let h' be an element of $\text{Symp}(N)_0$, we can lift h' to an element h'' of U_N , since h' is an element of $\text{Symp}(N)_0$, h'' commutes with the action of $\pi_1(N)$ on U_N . We denote by $\text{Symp}'(N)_0$, the group of symplectomorphisms of (U_N, ω') (where ω' is the symplectic form of U_N lifted to U_N by the covering map $p' : U_N \rightarrow N$) which commutes with the holonomy: this group is a covering of $\text{Symp}(N)_0$. Since the element h'' commutes with $\pi_1(N)$, it induces a gauge transformation on $V_{h_N}^1$ which covers the identity of N . The covering map $\hat{\text{Symp}}(N)_0 \rightarrow \text{Symp}'(N)_0$ induces an action of $\hat{\text{Symp}}(N)_0$ on $V_{h_N}^1$.

To study the action of $\text{Symp}(N)$ on the configuration D , we make the following assumption: let N_U be a fundamental domain of the action of $\pi_1(N)$ on U_N , we denote by D' the boundary of U_N , we suppose that the image of D' by p' is D . We obtain the following proposition:

Proposition. *The group $\hat{\text{Symp}}(N)_0$ acts naturally on $V_{h_N}^1$. The stabilizer of the image of $\hat{N} \times N_U - D'$ by the covering map $p'' : \hat{N} \times U_N \rightarrow V_{h_N}$ is $\text{Symp}(\hat{N}, N_U)_0$, the connected component of $\hat{\text{Symp}}(N)_0$ which fixes N_U .*

We suppose that the manifold V is endowed with a symplectic form ω_V such that the action of H on V is symplectic, and the maps $\phi_i : U_i \rightarrow V$ which define the (V, H) -structure of N are symplectic maps. This enables to define on $V_{h_N}^1$ the following symplectic structure: on the trivialization $U_i \times V$, we define the form Ω_i to be the product of the restriction of ω to U_i with $-\omega_V$. Since the action of H on V is symplectic, we deduce that the forms Ω_i glue together to define a form Ω on $V_{h_N}^1$. Since we can assume that the section l_{h_N} takes its values in $V_{h_N}^1$, we have the following proposition:

Proposition *The section l_{h_N} which defines the (V, H) -structure is a Lagrangian submanifold. The orbits of l_{h_N} under $\hat{\text{Symp}}(N)_0$ are also Lagrangian submanifolds, conversely every Lagrangian submanifold in the connected com-*

ponent of l_{h_N} in the space of Lagrangian submanifolds transverse to the vertical foliation of $V_{h_N}^1$ is the image of l_{h_N} by an element of $\hat{\text{Sym}}(N)_0$.

proof.

The section l_{h_N} is defined on U_i by $l_{h_N}^i(u) = (u, \phi_i(u))$. Since the maps ϕ_i are symplectic maps, we deduce that the section $l_{h_N}^i$ is a Lagrangian submanifold as is the section $\phi(l_{h_N}^i)$ defined by $\phi(l_{h_N}^i)(u) = (u, \phi \circ \phi_i(u))$.

Let d' be a Lagrangian submanifolds of $V_{h_N}^1$ transverse to the vertical foliation of $V_{h_N}^1$, d' is the image of the section d of $V_{h_N}^1$. Consider a trivialization $(U_i \times \phi_i(U_i), \phi_j \circ \phi_i^{-1})$ of $V_{h_N}^1$, we define the symplectomorphism ϕ_d of N whose restriction ϕ_d^i to U_i is defined for every element u of U_i by the projection of $d_i(u)$ by the covering map $p'' : U_N \rightarrow N$, where d_i is the restriction of d to U_i . The morphism ϕ_d is well-defined indeed, suppose that $u \in U_i \cap U_j$, then $d_j(u) = \phi_j \circ \phi_i^{-1} d_i(u)$. This implies that $\phi_d^i(u) = \phi_d^j(u)$. The section d is the image of l_{h_N} by an element $\hat{\phi}_d$ of $\hat{\text{Sym}}(N)$ above ϕ_d . If d' is in the connected component of l_{h_N} in the space of Lagrangian submanifolds transverse to the vertical foliation of $V_{h_N}^1$, we can suppose that ϕ_d is an element of $\text{Sym}(N)_0$.

Suppose that there exists a pseudo-complex structure J adapted to the symplectic form of $V_{h_N}^1$ such that the induced differentiable metric is complete, J enables to define an isomorphism of bundle between the cotangent bundle of l_{h_N} , $T^*l_{h_N}$ and Nl_{h_N} the orthogonal of the tangent bundle Tl_{h_N} of l_{h_N} in $V_{h_N}^1$. We consider the differentiable map $P_N : T^*l_{h_N} \rightarrow V_{h_N}^1$ defined as follows: identify $T^*l_{h_N}$ with the normal bundle of l_{h_N} in $V_{h_N}^1$ by the map which assigns to an element n_u of the fiber of the normal bundle at u the 1-form $i_{n_u}\Omega$. Let v_u be an element of the fiber of u in $T^*l_{h_N}$, $P_N(v_u) = \exp_u(v_u)$, where the exponential map is defined by the differentiable metric $\Omega(J, \cdot)$.

Theorem.

Suppose that P_N is a symplectomorphism which induces a one to one map between the Lagrangian submanifolds of $T^*l_{h_N}$ transverse to the vertical foliation of $T^*l_{h_N}$, and the Lagrangian submanifolds of $V_{h_N}^1$ transverse to the vertical foliation of $V_{h_N}^1$, then the orbit of l_{h_N} under $\hat{\text{Sym}}(N)_0$ is contractible.

proof.

Consider the cotangent bundle $T^*l_{h_N}$ endowed with the differential of the Liouville form. The map P_N is a symplectomorphism which send Lagrangian submanifolds of $T^*l_{h_N}$ transverse to the vertical foliation to Lagrangian submanifolds of $V_{h_N}^1$ transverse to the vertical foliation. Since the Lagrangian submanifolds of $T^*l_{h_N}$ transverse to the vertical foliation are one to one with closed 1-forms defined on l_{h_N} , we deduce that the orbit of l_{h_N} under $\hat{\text{Sym}}(N)_0$ is contractible.

Symplectomorphisms group of symplectic affine manifolds.

Let (N, ∇_N) be a complete compact n -dimensional affine manifold endowed with the parallel symplectic form ω . The bundle V_{h_N} is the quotient of $\mathbb{R}^n \times$

\mathbb{R}^n by the action of $\pi_1(N)$ which acts on the both factors by the holonomy representation. The principal bundle associated to V_{h_N} is the bundle of affine frames.

Theorem *Let N be an n -dimensional compact affine manifold, suppose that the affine structure of N is defined by a flat differentiable metric, and N is endowed with a symplectic form parallel in respect to the flat connection, then if $\text{Ham}(N, D)_0$ is contractible, then $\text{Ham}(N)_0$ is also contractible.*

proof.

The manifold N is the quotient of \mathbb{R}^n by a subgroup $\pi_1(N)$ whose linear part is contained in $O(n)$. The parallel symplectic form ω_0 is invariant by the holonomy and gives rise to the form ω of N . The parallel complex structure J_0 adapted to the flat metric of \mathbb{R}^n , and to ω_0 gives rise to the pseudo-complex structure J of (N, ω) . In this situation, $P_N : T^*l_{h_N} \rightarrow V_{h_N}$ is a symplectomorphism.

Let α be a closed 1-form defined on N , suppose that $P_N(\alpha)$ is transverse to the vertical foliation, then it defines an symplectomorphism $\hat{\phi}_\alpha$ of V_{h_N} which gives rise to a symplectomorphism ϕ_α of N , we can define $\phi_\alpha(D)$ the image of D by ϕ_α . Suppose that $P_N(\alpha)$ is not transverse to the vertical foliation, then the infinite dimensional Sard lemma implies the existence of a sequence of closed 1-forms $(\alpha_n)_{n \in \mathbb{N}}$, such that α_n is transverse to the vertical foliation, and α_n converges towards α . We define $\phi_\alpha(D)$ to be the limit of $\phi_{\alpha_n}(D)$. This make sense because the group of symplectomorphisms is C^0 -closed in the group of diffeomorphisms.

Let $N'(D)$ be the image of the space of closed 1-forms $C^1(\alpha)$ by the map Φ defined by $\Phi(\alpha) = \phi_\alpha(D)$, $N'(D)$ contains an open and dense subset of $N(D)$. Indeed, consider the subspace of image of closed 1-forms α such that α is transverse to the vertical foliation. This image is open, since the neighborhood of a symplectomorphism can be identified with a neighborhood of the zero section in the cotangent bundle, and the image of the set $N''(D)$ of closed 1-forms such that the image of their graph by P_N are transverse to the vertical foliation is dense in the space of graphs of symplectomorphisms this implies also that $N'(D)$ is closed in $N(D)$, thus $N'(D) = N(D)$.

The space of closed 1-forms $C'^1(N)$ which induces symplectomorphisms which fix D is a vector subspace since the components of D are affine submanifolds. To see this we can use a generating functions-like theory. Indeed, let α be a closed 1-form, the image of the lift of the graph to the universal cover of \hat{N} by P_N is the space of elements of $\mathbb{R}^n \times \mathbb{R}^n$ of the form $(u_1 + \alpha_1, \dots, u_n + \alpha_n, u_1 - \alpha_1, \dots, u_n - \alpha_n)$ where $(\alpha_1, \dots, \alpha_n)$ are the component of the lift $\hat{\alpha}$ of α to \mathbb{R}^n . The fact that ϕ_α preserves D implies that if $(u_1 + \alpha_1, \dots, u_n + \alpha_n)$ is an element of the subspace \hat{D} of \mathbb{R}^n over D , then $(u_1 - \alpha_1, \dots, u_n - \alpha_n)$ is also an element of \hat{D} . Let α, α' be closed 1-forms which define symplectomorphisms of N which preserves D , then if U is a connected component of \hat{D} in \mathbb{R}^n , then the element $(u_1 - \alpha_1, \dots, u_n - \alpha_n)$ is tangent to U if $(u_1 + \alpha_1, \dots, u_n + \alpha_n)$ in this situation, (u_1, \dots, u_n) and $(\alpha_1(u), \dots, \alpha_n(u))$ are tangent to U . This implies that $(\alpha_1 + \alpha'_1, \dots, \alpha_n + \alpha'_n)$ is tangent to U , and henceforth that if $\alpha + \alpha'$ induces a sym-

plectomorphism then the fact that $(u_1 - (\alpha_1 + \alpha'_1), \dots, u_n - (\alpha_n + \alpha'_n))$ is an element of U , if $(u_1 + \alpha_1 + \alpha'_1, \dots, u_n + \alpha_n + \alpha'_n)$ is an element of U implies that this symplectomorphism preserves D . We have a fibration $C'^1(N) \rightarrow C^1(N) \rightarrow N(D)$. Since $C^1(N)$ and $C'^1(N)$ are contractible, we deduce that $N(D)$ is contractible.

Corollary.

The universal cover $\widehat{Symp}(N)$ of the connected component of the group of symplectomorphisms $Symp(N)$ of a 4-dimensional compact manifold endowed with a flat metric, and a parallel symplectic structure ω is contractible.

Proof. N is the quotient of \mathbb{R}^4 by $\pi_1(N)$, the boundary D of this action is a configuration. We have a fibration $Symp(N, D)_0 \rightarrow Symp(N)_0 \rightarrow N(D)$, since $N(D)$ is contractible, we deduce that $Symp(N)_0$ has the homotopy type of $Symp(N, D)_0$. We have an exact sequence $Ham(\widehat{N}, D) \rightarrow \widehat{Symp}(N, D) \rightarrow L$, where L is the image of $\widehat{Symp}(N, D)$ by the flux. Since $Ham(\widehat{N}, D)$ is contractible, we deduce that $\widehat{Symp}(N, D)$ is contractible.

References.

1. Abreu, McDuff, Topology of the symplectomorphisms group of rational surfaces. J. of American Math Society.
2. Banyaga Sur la structure du groupe des diffeomorphismes qui preservent une forme symplectiques. Comment. Math. Helvetici.
3. Joseph Coffey, Symplectomorphisms groups and isotropic skeletons, Geometry and Topology.
4. Lalonde, Pinsonnault, The topology of the space of symplectic balls in rational 4-manifolds. Duke. Math. Journal.
5. Goldman, The geometric structures and varieties of representations.
6. Pinsonnault, Remarques sur le groupe des symplectomorphismes de l'eclatement de $S^2 \times S^2$. These Universite U.Q.A.M.
7. Ruan, Y. Symplectic topology and extremal rays, Geometry and functional analysis.
8. Tsemo, A. Homotopy of symplectomorphisms groups. In preparation.